Mathematics and its application in various complex problems: A Survey

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ABSTRACT
In this chapter we review the state-of-the-art in non-visual accessibility of mathematics. Making mathematics accessible is a significant challenge, due to its 2-dimensional, spatial nature and the inherently linear nature of speech and Braille displays. The relative arrangement of various mathematical symbols in the 2-dimensional print space is extensively exploited by standard mathematical notations to succinctly convey information to the sighted reader (e.g., square roots, exponentiations, fractions). The spatial layout implicitly encodes semantic information which is essential to the understanding of mathematical constructs, however, it poses a hurdle in making mathematics accessible in the two most traditional media adopted by visually impaired students, i.e., Braille and aural rendering. In both approaches, the information implicit in the spatial arrangements has to be made explicit, making accessing and understanding of mathematics by the visually impaired cumbersome. In this chapter, we consider various approaches proposed as well as in use for making mathematics accessible. We survey approaches based on Braille codes (e.g., the Nemeth and Marburg codes) as well as those based on aural rendering of mathematical expressions (such as AsTeR and MathGenie). We cover existing systems that are in use for making mathematics accessible, along with current research in the area.

Keywords: Mathematics, Accessibility, Visual Disabilities

1. Introduction

The study of mathematics is crucial in the preparation of students to enter careers in science, technology, engineering and related disciplines such as the social and behavioral sciences. For many sighted students, math education poses a serious roadblock in entering technical disciplines, which has a serious impact on our economic competitiveness and science-related capabilities. For the visually impaired student, the roadblock is even higher, due to the additional difficulties they have to face in accessing mathematics.

Presentation of written information to blind individuals has traditionally been accomplished through the use of Braille. For presenting text, while Braille may not have been an ideal solution, it has certainly been a satisfactory one. Traditional Braille utilizes a raised character set composed of six dots per character, which limits the character set to 64 possible characters. Even for simple text, this does not represent an adequate alphabet. To solve the problem, most Braille notations use multi-character representation. For example,

“A” (capital a) is represented in American standard Braille by a sequence of two characters: “,a”, i.e., the letter a preceded by a comma..

In recent years, there have proponents of an eight-dot system, which could then allow an alphabet of 256 unique characters (Schweikjardt 1998; Gardner 2005). For a variety of reasons, that resemble the “QWERTY” vs. the Dvorak[1] keyboard debate, the eight-dot systems have not gained much popularity. While the text Braille issue is complex, it pales in comparison to the difficulties in the representation of math in Braille. While text is linear in nature, anything but the simplest math is not normally represented linearly. Note that Braille itself is also linear. Thus, the multitude of mathematical operators and special symbols has to be simulated by various sequences of Braille characters. Additionally, the spatially arranged structures have to be linearized so that they can be represented in Braille (e.g., the square root operation). All these requirements make math learning and teaching very complex.

In 1990, the United States passed landmark legislation, the Americans with Disabilities Act, to directly address a broad range of problems. Similar legislations have been passed by many other countries. These laws have significantly improved accessibility for people with disabilities, but they have had marginal success with math and the visually impaired. Other legislation such as The Education Act of 1973, The Telecommunications Act of 1996 and the aggressive implementation of Section 508 of The Education Act of 1973, have resulted in many positive changes. Unfortunately, the math education of the visually impaired has not seen any noticeable improvement. In response, numerous projects have surfaced to bring closure to the problem. A simple Google search on the words "math blind," currently returns 1.4 million hits. The problem however is complex, and dual pronged: to make meaningful headway, the needs of both the students and their teachers have to be taken into account. Thus, any solution to the problem must make it easy for blind students to internalize ("visualize") mathematical expressions. Likewise, a good solution should not place undue burden on the teachers, in terms of preparing the material for the student, or having to learn substantial new material (e.g., learning a Braille Math notation). There have been several interesting projects over the years that have addressed these issues, and they will be discussed later in this chapter. While a generic solution has yet to be developed, progress is being made. With current inter disciplinary research projects, the underlying issues are being exposed by teaming mathematicians, cognitive psychologists, human computer interaction experts and the user community.

2. A Classification of Math Accessibility Approaches

The goal in making mathematics accessible to a blind student is to ensure that entire mathematical document is accessible. This means that not only individual mathematical expressions are accessible, but sequences of expressions that may arise, for example, in a proof, as well as text embedded between mathematical expressions are also accessible. The approaches to making Mathematics accessible can be classified into two types: static and dynamic.

· **Static Approaches:** the mathematical content is statically converted into a format that is reproducible using assistive devices (such as Braille refreshable displays and other tactile devices) or that can be printed on Braille paper. In these approaches, the document is mostly viewed as a *passive* entity (akin to a printed document presented to a sighted user), while the *active* component is represented by the user, who uses an assistive device (e.g.., a refreshable Braille display) to move around the document, reading parts, skipping other parts, backtracking through it, etc.

· **Dynamic Approaches:** the mathematical content is presented in a dynamic, interactive fashion. These approaches require a conversion process which allows the user to navigate the mathematical content in accordance with its mathematical structure. In this case, the document itself becomes an *active* component; by performing intelligent transformation of the document, its semantic structure is exposed and information overload on the user reduced.

Note that static approaches render mathematical expressions in Braille, while dynamic approaches render it using audio – alone or along with other traditional techniques, such as refreshable Braille. It should be noted that the two are complementary to each other. Experience indicates that just as the ability to read and write is important for sighted individuals (in addition to being able to hear things), the ability to write using Braille-based codes is likewise important for blind individuals as audio hearing is not enough, howsoever interactive it may be (National Library of Service 2000).

3. Static Approaches

3.1. Introduction

In the static approaches sophisticated, special Braille notations have been developed for mathematics. Virtually every national Braille notation also has at least one special version for math. These math codes attempt to present complex mathematical expressions in a way that blind individuals can follow them. These notations include the Nemeth Math code (Nemeth 1972) in use in US, Canada, New Zealand, Greece, and India, the Marburg code (Epheser, Pograniczna et al. 1992) in use in Germany and Austria, the French Math code (Commission Evolution du Braille Francais 2001), ItalBra (Biblioteca Italiana per Ciechi "Regina Margherita" O.N.L.U.S. 1998), the Italian Math code in use in Italy, and the British Math Braille code (Braille Authority of the United Kingdom 1987). Unfortunately, these notations introduce their own problems.

The Braille-based mathematical notations devised in various countries constitute a new language, and consequently must be learned by the students. Of course, this learning has to be supervised by teachers, who themselves need to be proficient in these notations. In the United States and many other countries, there is a paucity of math and science teachers (Lee 2005) and especially trained special education teachers capable of teaching both math and complex Braille notations (Gill 2006). With the broad acceptance of "mainstreaming," and the resulting demise of special education centers for the visually impaired, the problem has become even more difficult. This is because a Math teacher at a school, community college or school may come across one or two blind students over a span of few years, and thus does not have a very strong incentive to learn the Math codes. The situation is somewhat alleviated through engagement of special education teachers who know the Braille math code and help teachers with blind students in their classes on a demand basis.

There are several national codes that have been designed for encoding Mathematics in 6-dots Braille. We give an overview of these codes below. We also give an overview of assistive technology tools that have been developed to make the task of learning and teaching Mathematics for blind students and their teachers respectively.

3.2. 6-dots Braille

The 6-dot Braille was invented by Louis Braille in the 1820s. It uses 6 dots arranged in 3 rows of 2 dots each which are raised depending on the letter or symbol to be represented. There are 64 possible letters and symbols that can be represented by the 6-dot system (no raised dots represents a blank character, thus there are 63 letters/symbols where at least 1 dot is raised). The invention of Braille allowed blind individuals to read and write for the first time at speeds equal to or surpassing those of the sighted individuals in writing print text (Braille 1829). What Braille did was to provide an alphabet to allow blind individuals to read and write. This leap opened door to other possibilities, such as Braille-based code for Mathematics, Science, Music, etc. Braille-based codes for Mathematics started appearing in the mid to late 1900s: For example, the Nemeth code, designed by blind Mathematician Abraham Nemeth, was published in 1951, while the Spanish Math Braille code was developed in 1987.

3.3. Mathematics-specific Braille Extensions

The use of 6-dot Braille for the encoding of mathematical content has been widely criticized. The 6-dot format can only produce 63 different Braille cells (observe that in 6-dot Braille an unused cell or blank cell is implicitly considered as a space). As a consequence multiple sequences of Braille cells have to be used to encode distinct symbols, and many Braille cells have different meanings depending on the context they appear in. This type of designs have an unfortunate consequence – formats like Nemeth Braille become *context sensitive languages* (Hopcroft, Motwani et al. 2000), which are inherently hard and expensive to parse and translate.

For this reason, there have been many attempts made to expand Braille codes to 8-dot formats. 8-dot allows a larger number of Braille cells (256), simplifying the encoding of a larger set of symbols and create a better consistency with the 8-bit standard ASCII character set used on most computers. A number of official and semi-official general 8-dot Braille codes have been developed, mostly in Europe, but relatively little literature has been reproduced in any 8-dot Braille code. In the context of representation of mathematical content, two relevant proposals targeting 8-dot representation of mathematical content are DotsPlus (Gardner 2003) and LAMBDA (Edwards, McCartney et al. 2006).

DotsPlus Braille gracefully extends 6-dot grade 1 standard Braille, with the exception that most double cell characters of the 6-dot font are single cells in the 8-dot font. Capital letters, for example, are encoded with an extra dot (dot-7 position) on the left side of the row below the bottom of the standard 6-dot lower case letter cell. The novelties of DotsPlus include

· Most punctuation marks are not Braille but small graphic symbols; similarly, most of the characters from complex literature (e.g., mathematical symbols) are encoded as graphic symbols shaped similarly to the corresponding print symbols.

· Numbers are encoded in single cell; the digits from 1 to 9 are encoded as in the literary Braille number mode with an additional dot-6 (the digit 0 has a distinct encoding to avoid conflict with the letter 'w').

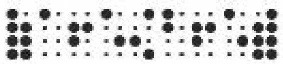
The outcome is a notation that is easier to learn and remember for the representation of mathematical content. Unusual symbols are encoded in a shape analogous to the print format, allowing visually impaired individuals to follow the same learning/remembering process as a sighted reader, and avoid having to memorize complex sequences of Braille cells. Furthermore, DotsPlus Braille reduces the context dependence of individual symbols in a Braille line, leading to the ability to print math in standard print format (including positional placements of symbols). Figure 1, drawn from (Gardner 2002), shows the print format and the DotsPlus format of a simple equation. The success of DotsPlus is also related to the ability to

ax^2 □bx □c □0

Figure 1: A simple equation in print format and DotsPlus format use the Tiger Tactile Graphics and Braille Embosser (Gardner, Ungier et al. 2006), which includes printer drivers for DotsPlus fonts.

The LAMBDA project introduced a linear math notation, the LAMBDA code (Edwards, McCartney et al. 2006; Fogarolo 2006; Schweikhardt, Bernareggi et al. 2006), largely inspired by the way mathematical content is encoded in MathML. Thus, the LAMBDA code plays the role of a markup language – i.e., it provides markers to denote special types of expressions and representation of common symbols – expressed in an 8-dot Braille notation, to provide linear encoding of mathematical content. The LAMBDA code is meant to be an internal source code for the representation of mathematics, and tools have been investigated to translate LAMBDA code to national Braille formats (e.g., Italian Braille) for output purposes.

For example, the representation of a fraction requires three markers: one denoting the beginning of the fraction, one representing the end of the numerator, and one representing the end of the fraction. For example, the fraction would be represented as <start fraction> a+1 <fraction symbol> b+1 <end fraction>. The three markers have different 8-dot Braille representations depending on the specific country; the above fraction, in UK 8-dot LAMBDA code, would appear as (Fogarolo, Bernareggi et al. 2005):



where  represents the <start fraction>,  represents the <end fraction>, and  represents the fraction symbol.

The use of 8-dot Braille code, with its 256 possible combinations of dots, implies that many markers and symbols require multiple Braille cells. The format structure adopted in the LAMBDA code tries to appeal to logical constructions to facilitate the interpretation and recollection; in particular, many symbols are constructed using prefixes, which identify the class the symbol belongs to. For example, Greek symbols are constructed by a fixed prefix followed by the Braille code for the corresponding Latin letter, e.g., the symbol □ is represented as:

As another example, the symbol □(set union) is encoded as

where the first cell is a prefix denoting set operations, and the second cell is the traditional encoding of '+'.

The LAMBDA code is supported by a sophisticated LAMBDA editor, which recognizes and handles the hierarchical structure of mathematical expression, automatically manages the different blocks composing it, and allowing different forms of visualization (e.g., it allows to hid the content of the blocks).

4. Dynamic Approaches
4.1. Foundations of Math Presentation
4.1.1. Cognitive Foundations
Relatively little work has been done in investigating the cognitive aspects of human interaction with mathematical content. The most relevant effort in this areas has been reported in (Gillan, Barraza et al. 2004). This work studies the perceptual and cognitive processes used by sighted individuals during equations reading; the ultimate goal is to understand these processes to the extent of being able to develop aural presentation mechanisms that provide visually impaired individuals with equivalent process capabilities. The investigation in (Gillan, Barraza et al. 2004) was conducted using "think aloud" protocols and eye-tracking devices, and it involved separate experiments aimed at measuring

1. The capability of recalling equations – the experiment consisted of exposure to different equations for varying period of times, followed by presentation of "distracting" screens.
2. The impact of knowledge of the structure of the equation – the experiment exposed the subjects to a preview of the structure of the equation, and measured whether this affected the time to solve the equation.

3. The extent to which subjects use a "chunking" approach in reading equation, i.e., they decompose the equation into chunks, defined by expressions within parentheses, solve such sub-expressions and store their outcome in memory.

The outcome of this study can be summarized as follows:

- the reading process is mostly a left-to-right process, moving one element at a time (similarly to the way standard text is read)
- an initial scan is typically performed to acquire the structure of the equation before proceeding to its detailed understanding
- readers backtrack very frequently when understanding/solving an equation
- readers tend to process operators and numbers more deeply than parentheses
- readers chunk together parts of an equation (especially the content of a sub-expression within parentheses) and process the chunk modularly; as a consequence of chunking, the readers tend to solve the equation hierarchically.

5. Open Problems and Perspectives for the Future

5.1. Localization Problems
The previous efforts on translation between mathematical formats for mathematics have already addressed the presence of significant differences in notations across national boundaries.

The differences go beyond the specific notations – as distinct approaches, traditions, and methodologies affect the way mathematics is seen and understood. This implies that approaches to presentation of mathematics should include localization components and customize the delivery to the specific national standards.

Relatively limited work has been conducted to enable presentation methodologies to apply localization to its output. The Universal Math Conversion library (Archambault, Fitzpatrick et al. 2004) has been developed with the specific intent of facilitating the inter-operation between Braille mathematical documents expressed using different national math Braille notations – including the development of a canonical subset of MathML specifically designed to facilitate the inter-conversion process (Archambault and Moço 2006). The LAMBDA project (Edwards, McCartney et al. 2006) lays the foundations of the design of its 8-dot code on understanding the peculiarities of the different national codes used in different European countries. LAMBDA defines an open-separator-close tag structure to linearly represent mathematical structures. In order to be actually usable, the tag structures have to be concretized according to the national conventions (this was done for each country participating in the LAMBDA consortium). The instantiation process requires the following components:

- The specification of all dot configurations for the symbols which are represented on one Braille cell (e.g. lower-/upper-case Latin letters etc.).
- The definition of the notations to be described. All the mathematical notations used in an educational curriculum have to be linearly described.
- The assignment to each tag of a name to be displayed in the list of tags in the mathematical editor. The name depends on the national languages.
- The assignment to each tag of at least two names to be used in the preparation of the speech output. They are necessary to process the string to be read by the speech synthesizer.

Each localization of LAMBDA is encoded in an XML structure, to enable the LAMBDA-related tools (e.g., the mathematical editor) to easily retrieve information and set the local working environment.

5.2. Integration of Components

The previous discussion highlights the complexity of providing non-visual access to mathematical content. In particular, it is clear that a number of dependent issues have to be effectively addressed, ranging from inter-operation between representation formats, adoption of different assistive devices, and modalities of presentation/navigation. These issues are strongly dependent on each others, forcing researchers to address the whole spectrum of options and issues. This has led in recent years to the development of wide-breadth projects that span a variety of aspects of accessibility of mathematical content in an integrated fashion. Two notable efforts in this direction are represented by the LAMBDA Project[5] and the International Universal Mathematics Accessibility group (iGroup UMA)[6].

The LAMBDA project was funded by the European Union with the objective of creating an integrated system for both writing and reading mathematical content for the benefit of blind students. The strategic structure of the project relies on the development of an editor, to write mathematical expressions in a linear way, capable of interacting with different assistive devices, and of a linear mathematical code, designed to be inter-operable with various existing formats for mathematics (including formats used by advanced systems for the manipulation of mathematics, e.g., Mathematica). The international collaboration involves teams from the U.K., Italy, Germany, Portugal, Greece, Spain, and France.

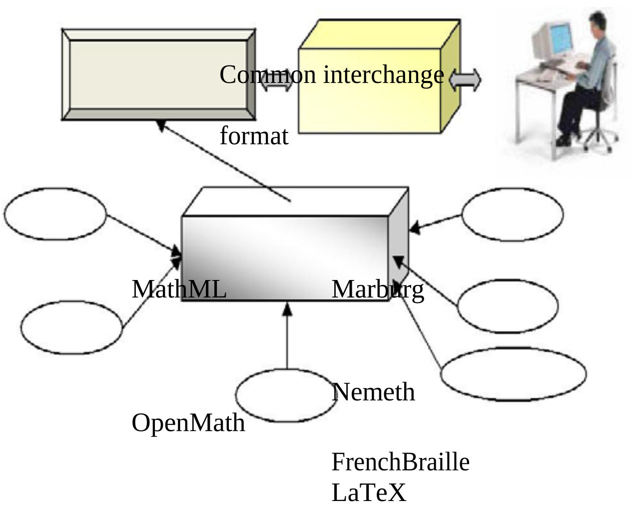


Figure 4: System Organization.

The iGroup UMA (Karshmer, Gupta et al. 2004) is an international cooperation, involving researchers from Ireland, Austria, France, Japan, and the U.S.A. The goal of the cooperation is to develop tools to enhance accessibility of mathematical content across different representation formats (e.g., Braille and digital formats), across National styles and conventions, and across different levels of visual capabilities. Figure 4 shows the overall organization of the project. The project provides a uniform inter-conversion platform which allows two-directional translation between a wide range of formats for the representation of mathematics. The inter-conversion platform relies on the use of a common interchange format (MathML in the case of the current version of the project) which is used as a bridge between any pair of formats considered. The project provides also tools for the rendering and navigation of mathematical content – currently this component is realized within the previously described MathGenie.

6. Conclusion

The problems associated with teaching mathematics to the visually impaired are long standing and difficult problems. With the advent of the reasonably inexpensive computer, many partial solutions have been put forward, as described in the body of this work. Each has added knowledge required to make incremental advances in the state of the art.

Additionally, the problems associated with math and blindness are now being studied from an interdisciplinary perspective, adding needed fundamental understanding to the design and development process. The data generated by these efforts have been invaluable to the understanding of the problem©s nature and potential solution. Finally, and most importantly, is the advent of real international cooperation in understanding the dimensions of the problem and their potential solution. As most governmental units do not view the problem as one of great import, and the obvious lack of potential wealth to be derived from solving the problem, the international research community has banded together to make essential breakthroughs in the past decade. This effort is to be applauded and encouraged.

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