# The Classical Laplace Transform and its q- Image of the Wright type Hyper geometric function

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**ABSTRACT**
Although the q-Laplace transform is the q-image of classical Laplace but q- Calculus has served as a bridge between mathematics and physics in particular in case of quantum physics. The present paper deals with one of the important result of q-Laplace transform of Wright type hyper geometric function in terms of well-known Fox's H-function. Also some special cases have been discussed in the lost section of the paper.

**Key Words and Phrases:** classical Laplace transform, q-image of Laplace transform, Wright type Function.

## 1. Introduction

The special functions owe their existence to the quest of researchers to find solutions of certain differential equations which occurred as mathematical models of well-known problems in physics. Since each definition was made to meet an exigency, all these functions are termed as special functions. The history of special functions is closely tied to the problems of terrestrial and celestial mechanics that were solved in the eighteenth and beginning of nineteenth centuries. This includes the boundary value problems of electromagnetism and heat in the nineteenth century and Eigenvalue problems of quantum mechanics in the first part of twentieth century .In these cases the common problem was to solve an ordinary or partial differential equation emerging due to mathematical formulation and modelling. Thus, special functions are mathematical functions catering to special needs in the field of physics, engineering, biology and other inter-disciplinary areas. This explains the adjective "special" prefixed to these functions. They form a class of well documented functions with extensive literature. The development of the theory of special functions and its applications went hand in hand. The enormous growth in the volume of research work on special functions witnessed during 18th and 19th century compelled the researches to think in terms of their unification. Their efforts in this direction were rewarding and it was found that a large number of special functions can be put in the form of generalized hyper geometric functions or their limiting cases

can be employed to arrive at the corresponding result for these special functions.

**Mathematical Preliminaries:**

**Fox's H – Function:** Fox has defined H-function in terms of a general Mellin-Barnes type integral. He also investigated the most general Fourier kernel associated with the H- function and obtained the asymptotic expansions of the kernel for large values of the argument. Fox has also derived theorems about the H- function as asymmetric Fourier kernel and established certain operational properties for this function.

The H – function is defined by Fox [1] as follows

$$H(z) = H_{p,q}^{m,n}\left[\left(z\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right)\right] = \frac{1}{2\pi i}\int_L \varphi(s) z^s ds$$

Where $\varphi(s) = \dfrac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}$

**Wright type Hyper geometric function:** The generalized form of the hypergeometric function has been investigated by Dotsenko [11], Malovichko [12] and one of the special case is considered by Dotsenko [11] as

$${}_2R_1^{\omega,\mu}(z) = {}_2R_1(a,b,c,\omega,\mu,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma\left(b+\frac{\omega}{\mu}n\right)}{\Gamma\left(c+\frac{\omega}{\mu}n\right)}\frac{z^n}{n!}$$

And its integral representation expressed as

$$2R_1^{\omega,\mu}(z) = \frac{\Gamma(c)\mu}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{\mu b-1}(1-t)^{c-b-1}(1-zt^{\omega})^{c-b-1}dt,$$

Where Re(c)$> Re(b) > 0. This$ is the analogue of Euler's formula for Gauss's hyper geometric functions [10].In 2001 Virchenkoetal [9] defined the said Wright type Hyper geometric function by taking $\frac{\omega}{\mu} = \tau > 0$ inabove equation as

$$2R_1(z) = 2R_1(a, b, c,,z) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}\frac{z^k}{k!}; \tau > 0, |z| < 1.$$

If $\tau$=1, then () reduces to Gauss's hyper geometric function.

**Classical Laplace transform:** suppose F(t) is a real valued function defined over the interval $(0, \infty)$. The Laplace transform of F(t) is defined by

$$L[F(t)] = \int_0^{\infty} e^{-st} f(t)\, dt. \qquad (1.1)$$

Or $f(s) = \int_0^{\infty} {}^{e-st} f(t) d(t)$

The Laplace transform is said to exist if the integral (1.1) is convergent for some values of s.

**Classical Fourier transform:** Iff(x) be a function defined on $(-\infty, \infty)$ uniformly continuous in finite interval and $\int_0^{\infty} \| f(x) \| d(x)$ converges. The Fourier transform is defined by

$F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} d(x)$, Where $e^{isx}$ is said to be kernel of the Fourier transform.

**q-image of Laplace transform:** Hahn [6] defined the q-image of classical Laplace transform

$$L[F(t)] = \int_0^{\infty} e^{-st} f(t)\, dt. \qquad (1.2)$$

The Laplace transform of the power function is defined as

$$L(t^{\mu}) = \frac{\Gamma(\mu+1)}{s^{\mu+1}} \qquad (1.3)$$

The q- Laplace transform of the power function is defined as
By means of the following q-integrals

$$L_q f(s) = \int_0^{\infty} e_q^{-sx} f(x) d(x), \mathrm{Re}(s) \supset 0. \qquad (1.4)$$

The q-Laplace transform of the power function is defined as in [10 & 11]

$$L_q(t^{\mu}) = \frac{\Gamma_q(\mu+1)(1-q)^{\mu}}{s^{\mu+1}} \qquad (1.5)$$

The Single parameter Mittag-Leffler Function is defined as follows.

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\alpha n)}, \text{ for } \alpha \in C, R(\alpha) \supset 0 \qquad (1.6)$$

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta+\alpha n)}, \text{ for } \alpha \in C, R(\alpha) \supset 0 \qquad (1.7)$$

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\beta+\alpha n)}\frac{1}{n!}, \text{ for } \alpha, \beta, \gamma \in C, R(\alpha) \supset 0 \qquad (1.8)$$

Where, $(\gamma)_n = \gamma(\gamma+1)(\gamma+2)(\gamma+3)...$

And $(\gamma)_0 = 1$

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$$L({}_2R_1(a,b;c;\tau;z)) = \frac{1}{s}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\middle| S^{-1}\right]$$

**Proof:** The classical Laplace transform of Wright type hypergeometric Function in terms of Fox's H – function is given by

$$L({}_2R_1(a,b;c;\tau;z)) = L\left(\frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{(a)_k\Gamma(b+\tau k)}{\Gamma(c+\tau k)k!}z^k\right) \qquad (2.1)$$

Since, $(\gamma)_n = \dfrac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$

From equation (2.1) we have,

$$L({}_2R_1(a,b;c;\tau;z)) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(a)\Gamma(c+\tau k)k!}L(z^k)$$

We know that the Laplace transform of the power function is,

$$L(t^{\mu}) = \frac{\Gamma(\mu+1)}{s^{\mu+1}}$$

$$\textit{Therefore} \quad L({}_2R_1(a,b;c;\tau;z)) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)k!}\frac{\Gamma(k+1)}{s^{k+1}}$$

$$\textit{Or} \quad L({}_2R_1(a,b;c;\tau;z)) = \frac{\Gamma(c)}{s\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)\Gamma(k+1)}\frac{\Gamma(k+1)}{s^{k}}$$

$$\textit{Or} \quad = \frac{\Gamma(c)}{s\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+\tau k)}{\Gamma(c+\tau k)}s^{-k}$$

$$\textit{Or} \quad = \frac{\Gamma(c)}{s\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(1-(1-a)+k)\Gamma(1-(1-b)+\tau k)}{\Gamma(1-(1-c)+\tau k)}s^{-k}$$

$$\textit{Or} \quad L({}_2R_1(a,b;c;\tau;z)) = \frac{1}{s}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\middle| S^{-1}\right]$$

This is the proof of theorem .

**Observations:**

**(1.1):** if $\tau = 1$ then from above theorem

$$L({}_2R_1(a,b;c;z)) = \frac{1}{s}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,1)\\(0,1),(1-c,1)\end{matrix}\Big| S^{-1}\right]$$

**Next Section :**

In this section of paper, the authors have derived the q-image Laplace transform of basic analogue Wright type hypergeometric Function in terms of Fox's q-H – function which is given by

**Theorem 2:** The q- Laplace transform of q-Wright type hypergeometric Function in terms of q-H – function is given by

$$L_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{1}{s}\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(1-b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\Big| S^{-1};q\right]$$

Proof: For $q > 0$, the q-image of Laplace transform of q-type of Wright type hypergeometric Function in terms of basic analogue of H – function is given by

$$L_q({}_2R_1(a,b;c;\tau;q;z)) = L\left(\frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(a;q)_k\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}z^k\right)$$

$$Or \qquad L_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{(a;q)_k\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}L\left(z^k\right)$$

Since, $(\gamma;q)_n = \frac{\Gamma_q(\gamma+n)}{\Gamma_q(\gamma)}$

and the q-Laplace transform of the power function defined in [10 & 11] as,

$$L_q(t^{\mu}) = \frac{\Gamma_q(\mu+1)(1-q)^{\mu}}{s^{\mu+1}}$$

Also, $(1-q)^{\alpha-1}\Gamma_q(\alpha) = (q;q)_{\alpha-1}$

$$Or \quad L_q({}_2R_1(a,b;c;\tau;q;z)) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}\frac{\Gamma_q(k+1)(1-q)^k}{s^{k+1}}$$

$$Or \qquad = \frac{\Gamma_q(c)}{s\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)(q;q)_k}\frac{(q;q)_k}{s^k}$$

$$= \frac{\Gamma_q(c)}{s\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(a+k)\Gamma_q(b+\tau k)}{\Gamma_q(c+\tau k)}\frac{1}{s^k}$$

$$= \frac{\Gamma_q(c)}{s\Gamma_q(a)\Gamma_q(b)}\sum_{k=0}^{\infty}\frac{\Gamma_q(1-(1-a)+k)\Gamma_q(1-(1-b)+\tau k)}{\Gamma_q(1-(1-c)+\tau k)}s^{-k}$$

$$= \frac{1}{s}\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(1-b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\Big| S^{-1};q\right]$$

Hence the theorem.

**Observations: (2.1):** if $\tau = 1$ then from above theorem

$$L_q({}_2R_1(a,b;c;q;z)) = \frac{1}{s}\frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(1-b,1)\\(0,1),(1-c,1)\end{matrix}\middle| S^{-1};q\right]$$

**Special cases:** Taking q=1, we get following as special cases of theorem (2)

$$L({}_2R_1(a,b;c;\tau;z)) = \frac{1}{s}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,\tau)\\(0,1),(1-c,\tau)\end{matrix}\middle| S^{-1}\right]$$

if $\tau = 1$ then from above theorem

$$L({}_2R_1(a,b;c;z)) = \frac{1}{s}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} H_{2,1}^{1,2}\left[\begin{matrix}(1-a,1),(b,1)\\(0,1),(1-c,1)\end{matrix}\middle| S^{-1}\right]$$

**References**


[1] Fox, C (1965), The G and H-functions as symmetrical Fourier Kernels. Trans. Amer. Math. Soc. 98, 395-429.

[2] Mittag-Leffler. G.M(1903), Sur la nouvelle fonction $E_\alpha$C. R. Acad. Sci. Paris 137,554-558.

[3] Wright, E.M (1935), The asymptotic expansion of generalized hypergeometric function. J. London Math. Soc, 10, 286-293.

[4] Hartley, T.T and Lorenzo, C.F (1998), A solution to the fundamental linear fractional order differential equation, NASA\TP.

[5] S.L.Kalla, L.Galue, R.N.Kalia, Generalized fractional calculus based upon composition of some basic operator,in: Recent Advancein fractional calculus,Global publ.co.USA,1993 pp.145-178

[6] S.L.Kalla, L.Galue,H.M.Srivastava ,further results on an H-functiongeneralized fractional calculus,J.Frac.Calcu.1993.89-102

[7] V.Kiryakova, the special functions of fractional calculus as generalized fractional calculus operators of some basic functions, J.Comput. Appl.Math.59(2010)1128-1141.

[8] V.Kiryakova. Multiple (Multi index) Mittag-Leffler functions and relations to generalized fractional calculus. J.Comput.Appl.Math.118(2000)214-259.

[9] N.Virchenko, S.L.Kalla and A,Al-zamel,''Some results on generalized hyper geometric functions'',Vol.12, no.1,2001,pp.89-100.

[10] A.Erdelyi,etal ''Higher transcendental functions,'' McGaw-Hill, New York,1953-1954.

[11] M.Dotsenko' On Some applications of Wright's hyper geometric functions'', Comptes Rendusdel, Accademie Bulgare des sciences, vol.44,1991,pp.13-16.

[12] V.Malovichko, ''On generalized hyper geometric functions and some integral operators'' Mathematcal physics, vol.19,1976.pp,99-103.