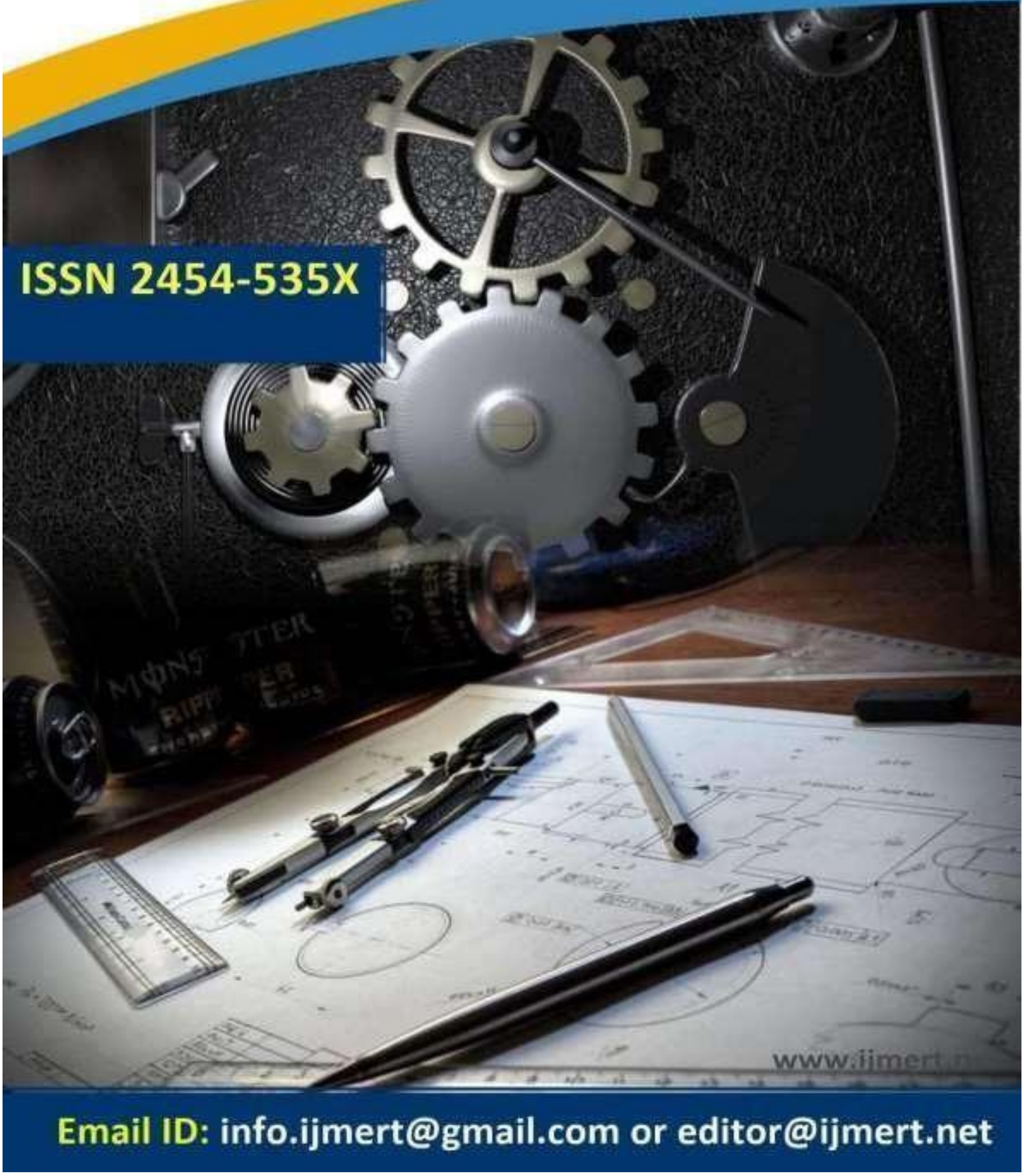
International Journal of
Mechanical Engineering Research and Technology

ISSN 2454-535X



Email ID: info.ijmert@gmail.com or editor@ijmert.net

Int. J. Mech. Eng. Res. & Tech 2017

International Journal of Mechanical Engineering Research and Technology

ISSN 2454 – 5X

http://www.ijmert.com
Vol. 15 Issue 4, Dec 2023

Principle of Divisibility for Discrete Monoids

K.Sushma [1], P.Naresh Kumar [2]
Assistant Professor [1], Associate Professor [2],
Department of Mathematics,
BRILLIANT GRAMMAR SCHOOL EDUCATIONAL SOCIETY'S GROUP OF INSTITUTIONS-INTEGRATED CAMPUS
Abdullapurmet (V), Abdullapurmet (M), R.R Dt. Hyderabad.

Abstract :

A concrete monoid over a category C is a subset of the endomorphisms of an object of C containing the identity and closed under composition To contrast an abstract monoid is just a one object category.

There is a natural notion of division between concrete monoids distinct from the usual division of abstract monoids This concrete division is identified via two examples and then defined giving rise to a bicategory of concrete monoids over C whose arrows are concrete divisions The Poincare classes of the arrows of this bicategory are found to have a simple and appealing characterization allowing us to define a category of concrete monoids over C .

These definitions are illustrated with examples from the theories of semigroups, matrices, vines and automata With the aid of these definitions, we make functorial the well known constructions of the action monoid of an automaton, and the endomorphism monoid of an object of a category.

Introduction

It is always a delicate matter to discuss mathematics informally,but in questions of motivation it often becomes necessary Therefore we begin with an informal account of the motivation for this work and ask the reader to bear in mind that when we use expressions such as ‘ abstract’ or ‘concrete’ or ‘interpretation’ we do not intend them yet to have a technical meaning This paper is based on the premise that there is a natural distinction between the property of being abstract or concrete To illustrate, one might say that an abstract set is one in which the elements have no particular interpretation, whereas a concrete set is one in which each element is to be interpreted as something For example: a set with three elements is an abstract set while a set of three oranges is a concrete set each of its elements is to be interpreted as a particular orange real or imaginary While it may seem churlish to make this distinction in the case that the set is the set of elements of a monoid.we give a number of examples which showthat the natural structural relationships between monoids whose sets of elements are abstract sets are different from the natural structural relationships between monoids whose sets of elements are concrete sets .

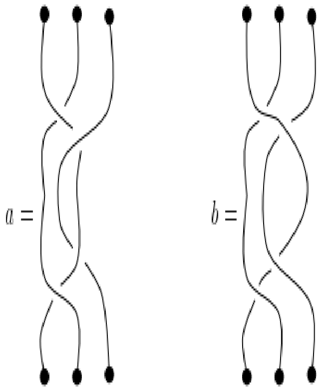
The primary examples of structural relationships between abstract monoids are the embedding and the quotient These are both readily (and often generalized to division crudely speaking) the monoid B divides the monoid A if B can be viewed as a submonoid of A when one ignores some of the structure of A (Formally, B divides A if there is asubmonoid C of A with B a quotient of C)Birkhoff’s Variety Theorem for abstract algebras. Reiter Mann’s Theorem for finite algebras, and the Krohn-Rhodes Theorem all attest to the fundamental importance of the division relationship.

In the following examples, we show that when the structure of the ‘things’ which make up the set of elements of a concrete monoid are taken into account the relationship of division is strengthened This stronger notion of division for concrete monoids which respects the structure of the elements of the monoid will be called a concrete division and this paper is devoted to finding a general definition for this stronger notion In particular, we use concrete division to construct a very simply defined morphism between concrete monoids which has an associative composition and hence we derive a category of concrete monoids.

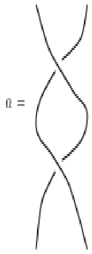
The purpose of this paper is threefold.

- *To give a general mathematical setting for the ideas discussed above and illustrated by the examples below. This formalization will hopefully account for all particular instances of this type of construction.*
- *To convince the reader (particularly the semigroup theorist) that a good general question when studying concrete monoids is "What are the concrete divisors of this monoid".*
- *To put forth the possibility that the category of concrete monoids will be useful in making functorial various constructions of concrete monoids from other mathematical objects.*

Example 1.1 [A concrete quotient] Consider the concrete monoid M whose elements are braids generated by the elements a and b depicted below.



Then removing the third string gives the *concrete quotient* Q generated by the single braid α:



with a quotient map defined by $a, b \mapsto \alpha$.

In the example above, since M is a submonoid of B_3, the braid group on 3 strings, the concrete quotient Q of M also gives rise to an obvious concrete division of B_3 by Q. Another example of a concrete division in the case that the monoids are transformation monoids will convince the reader of the diversity of situations in which constructions of this type arise.

Example 1.2 [A concrete division] This example concerns the case when the elements of the monoid are endofunctions of a set. (That is, the monoid is a *transformation monoid*.)

A monoid A of transformations of a set X will be represented as a table — each row of the table represents an element of the monoid and each column represents an element of the set. The ith entry in the row corresponding to $a \in A$ is $a(i)$, the value of a at i. The top row represents the identity element.

$$\text{Consider the transformation monoids } A = \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 3 & 3 \\ 2 & 2 & 3 & 4 \\ \hline 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \\ \hline 3 & 3 & 2 & 2 \\ 3 & 3 & 2 & 3 \\ 3 & 3 & 2 & 4 \end{array} \text{ and } B = \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 2 \end{array}.$$

We find B by 'filtering' A — by first deleting some of the rows of t
(not the top row), in such a way that the set of remaining rows i
under composition, and then deleting some of the columns, ensuring
remaining entries of the table consist only of elements whose colum
not been deleted. (In the table above, we have deleted column 4 a
5–7.) In short, we have found a submonoid N of A and a subset Y of

that for each $n \in N, y \in Y$, $n(y) \in Y$. To make the remaining part of the table into the transformation monoid B, we collapse identical rows. We now say that B_Y is a *concrete divisor* of the monoid A_X.

If this process is repeated with B to give a transformation monoid C, then C may be derived from A by a single step of deleting rows and columns and collapsing identical rows. This defines a composition of concrete divisions.

As an example of where these constructions arise, the concrete quotient construction between permutation groups is employed in algorithms to find the chief series and the sylow subgroups of a permutation group. The problem is solved for several concrete quotients and then the answers are "sewn together" to give an answer for the original permutation group [4], [5].

The first part of this paper is devoted to making this notion of a concrete division precise, and to defining the category **Conc**(**C**) whose objects are concrete monoids over **C** and whose arrows are equivalence classes of concrete divisions (in the example above, there would be an arrow from B to A).

The rather forbidding length of the following formalization is due to the fact that, being of general mathematical interest, the development has been made entirely self contained and assumes no prior knowledge of category theory beyond basic definitions and concepts.

Part A – Theory

This part is a systematic approach to the definition of the category **Conc**(**C**) of concrete monoids over **C**. In Section 2, we recall the definition of a bicategory and illustrate it in Section 3 with some relevant examples.

The category **Reps**(**C**) whose objects are monoid representations over **C** is defined in Section 4, and several useful theorems are proved. We take a detour in Section 5 to explain how the well established notions of *morphism*, *relational morphism* and *division* of monoid representations over **Set** fit into the framework constructed so far.

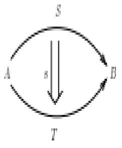
Finally, in Section 6 we study the bicategory of concrete divisions, whose objects are concrete monoids and whose arrows are the concrete divisions described in the introduction. Through a thorough understanding of this bicategory, we derive the category **Conc**(**C**) which is applied to various diverse situations in Part B.

2 Bicategories

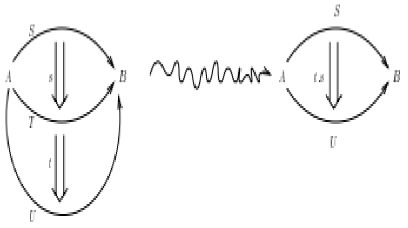
In his seminal work [3] Bénabou defines a bicategory. For completeness repeat this definition and an example almost verbatim. A *bicategory* determined by the following data:

B–I A set $Ob(\mathsf{S})$ called the set of *objects* of S.

B–II For each pair (A, B) of objects of S, a category $\mathsf{S}(A, B)$. An object S of $\mathsf{S}(A, B)$ is called an *arrow* of S and written $A \overset{S}{\to} B$. If S and T are objects of $\mathsf{S}(A, B)$, an arrow s of $\mathsf{S}(A, B)$ between S and T is called a 2-*cell* of S and will be written $S \overset{s}{\Rightarrow} T$ or more usually

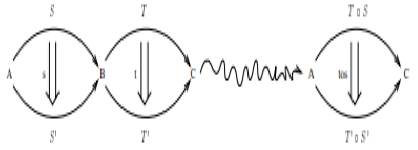


Composition in the category $\mathsf{S}(A, B)$ will thus correspond to the pasting



which is referred to as *vertical* composition of 2-cells.

B–III For each triple (A, B, C) of objects of S, a functor $c_{A,B,C}$: $\mathsf{S}(A, B) \times \mathsf{S}(B, C) \to \mathsf{S}(A, C)$ which is called the *composition functor* of S. For $A \overset{S}{\to} B$ and $B \overset{T}{\to} C$, write $T \circ S$ for $c_{A,B,C}(S,T)$. If $S \overset{s}{\Rightarrow} S'$ is an arrow of $\mathsf{S}(A, B)$ and $T \overset{t}{\Rightarrow} T'$ is an arrow of $\mathsf{S}(B, C)$ then write $t \circ s$ for $c_{A,B,C}(s, t)$. Then the composition functor corresponds to the pasting



which is referred to as *horizontal* composition of 2-cells.

B–IV For each object A of S an object I_A of $\mathsf{S}(A, A)$ called the *identity arrow* of A. The identity morphism of I_A in $\mathsf{S}(A, A)$ is denoted i_A and called the *identity* 2-*cell* of A.

B–V For each quadruple (A, B, C, D) of objects of S, a natural isomorphism $a_{A,B,C,D}$ between the two functors $c_{A,B,D} \cdot (id \times c_{B,C,D})$ and $c_{A,C,D} \cdot (c_{A,B,C} \times id)$ from $\mathsf{S}(A, B) \times \mathsf{S}(B, C) \times \mathsf{S}(C, D)$ to $\mathsf{S}(A, D)$. In particular, if $(S, T, U) \in \mathsf{S}(A, B) \times \mathsf{S}(B, C) \times \mathsf{S}(C, D)$ this gives a component isomorphism between $U \circ (T \circ S)$ and $(U \circ T) \circ S$.

B–VI For each pair (A, B) of objects of S, natural isomorphisms $l_{A,B}$ and $r_{A,B}$, called the *left* and *right* identities:

$$\begin{tikzcd}
1 \times \mathsf{S}(A,B) \arrow[rr, "\iota_A \times id"] \arrow[dr, "\pi_2"', ""{name=P}] & & \mathsf{S}(A,A) \times \mathsf{S}(A,B) \arrow[dl, "c_{A,A,B}", ""'{name=C}] \\
& \mathsf{S}(A,B) &
\arrow[Rightarrow, from=P, to=C, "l_{A,B}"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{S}(A,B) \times 1 \arrow[rr, "id \times \iota_B"] \arrow[dr, "\pi_1"', ""{name=P}] & & \mathsf{S}(A,B) \times \mathsf{S}(B,B) \arrow[dl, "c_{A,B,B}", ""'{name=C}] \\
& \mathsf{S}(A,B) &
\arrow[Rightarrow, from=P, to=C, "r_{A,B}"]
\end{tikzcd}$$

Here we write 1 for the trivial category and $1 \overset{\iota_A}{\to} \mathsf{S}(A,A)$ for the unique functor with value I_A. For each $S \in \mathsf{S}(A,B)$, the left and right identities give rise to component isomorphisms $I_A \circ S \cong S$ and $S \circ I_B \cong S$.

B–VII The data above are also required to satisfy the following *coherence conditions:*

(1) If (V,U,T,S) is an object of $\mathsf{S}(A,B) \times \mathsf{S}(B,C) \times \mathsf{S}(C,D) \times \mathsf{S}(D,E)$ then the following diagram commutes:

$$\begin{tikzcd}
((S \circ T) \circ U) \circ V \arrow[rr, "a(S,T,U) \circ id"] \arrow[d, "a(S \circ T,U,V)"'] & & (S \circ (T \circ U)) \circ V \arrow[d, "a(S,T \circ U,V)"] \\
(S \circ T) \circ (U \circ V) \arrow[dr, "a(S,T,U \circ V)"'] & & S \circ ((T \circ U) \circ V) \arrow[dl, "id \circ a(T,U,V)"] \\
& S \circ (T \circ (U \circ V)) &
\end{tikzcd}$$

This condition is known as *associativity coherence.*

(2) The following diagram commutes:

$$\begin{tikzcd}
(S \circ I_B) \circ T \arrow[rr, "a_{A,B,B,C}(S,I_B,T)"] \arrow[dr, "r_{A,B}(S) \circ id"'] & & S \circ (I_B \circ T) \arrow[dl, "id \circ l_{B,C}(T)"] \\
& S \circ T &
\end{tikzcd}$$

This condition is known as *identity coherence.*

2.1 From bicategories to categories

A bicategory is not necessarily a category — composition of arrows is o associative up to isomorphism. However, for any bicategory S, there are t closely related categories which capture much of the structure of S — *classifying category* and the *Poincaré category* of S [3].

The *classifying category* of S is the easier to define. Its objects are objects of S and its arrows are isomorphism classes of arrows of S. If S(A, B) and $T \in$ S(B, C) and if $[S]$ denotes the isomorphism class of S, t the composition of $[S]$ and $[T]$ in the classifying category of S is defined be $[T \circ S]$. The associative law now holds for composition of arrows, since axiom B-V above, there is a 2-cell isomorphism from $(S \circ T) \circ U$ to $S \circ (T \circ$

To define the Poincaré category, recall from [16, page 86] that a categ C is said to be *connected* if for any two objects A, B of C there is a fi sequence $A = A_1, A_2, \ldots, A_n = B$ of objects of C, and for each $i < n$, arrow $A_i \to A_{i+1}$ or an arrow $A_i \leftarrow A_{i+1}$. In [16, page 88, Ex. 7] it is sta that every category is the disjoint union of connected categories, called *connected components*. We may define an equivalence relation on the obje of a category by $A \sim_\pi B$ if A and B belong to the same connected compone

The objects of the *Poincaré category* are the objects of S and its arr are $\sim_\pi$ equivalence classes of arrows of S. Composition is defined as in classifying category. That this is a category follows by induction from B-I

3 Examples of Bicategories

Let A and B be objects in a category **C**. A *span* from A to B is a triple (f, S, g) where $A \xleftarrow{f} S \xrightarrow{g} B$ is a diagram of **C**. One of the many examples of a bicategory cited in [3] is Span(**C**), the bicategory of spans in **C**. Let **C** be any category with pullbacks, together with some extra structure – namely, a chosen pullback for every diagram of the form $\cdot \to \cdot \leftarrow \cdot$. We now define the bicategory Span(**C**) of spans of **C** with respect to this structure. Since a pullback is only unique up to isomorphism, another choice of pullbacks would give another bicategory which is isomorphic to the first.

S–I The objects of Span(**C**) are the objects of **C**.

S–II For each pair (A, B) of objects of **C**, the category Span(A, B) has as objects spans (f, S, g). An arrow of Span(A, B) from $A \xleftarrow{f} S \xrightarrow{g} B$ to $A \xleftarrow{f'} S' \xrightarrow{g'} B$ is an arrow $h : S \to S'$ such that $f' \cdot h = f$ and $g' \cdot h = g$. Composition in Span(A, B) is inherited from **C**.

S–III The composition functor is defined on arrows by taking the chosen pullback as in the following diagram:

$$\begin{tikzcd}
A & & B & & C \\
& S \arrow[ul, "f"] \arrow[ur, "g"'] & & T \arrow[ul, "h"] \arrow[ur, "k"'] & \\
& & S \times_B T \arrow[ul, "\pi_S"] \arrow[ur, "\pi_T"'] & &
\end{tikzcd}$$

and then setting $(h, T, k) \circ (f, S, g) = (f\pi_S, S \times_B T, k\pi_T)$. Horizontal composition of 2-cells is performed using the universal property of pullbacks.

The remaining details of this definition are left to the reader.

3.1 *Bicategories from a regular category:* Rel(C), Rel*(C) *and* Div(C).

When C is a regular category (a notion we will define presently), it is possible to construct bicategories Rel(C), Rel*(C) and Div(C) whose arrows are, respectively, relations, totally defined relations, and divisions between objects of C.

3.1.1 *Regular categories*

Fix some category C. Following [13] we define a *strong epimorphism* of C to be an epimorphism f such that if

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

is any commuting diagram with i monomorphic, then there is a (necessarily unique) w, called the *fill-in*, making the following diagram commute.

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \arrow[dl, "w"'] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

We repeat here some useful facts about strong epimorphisms which are taken from [13].

Proposition 3.1 *Strong epimorphisms are closed under composition and right division.* □

Proposition 3.2 *An arrow which is both a monomorphism and strong epimorphism is an isomorphism.* □

If f factors as ip where i is a monomorphism and p is a strong epimorphism, then we refer to ip as a *canonical factorization* of f.

Proposition 3.3 *Canonical factorizations are essentially unique. That is to say, if $f = ip = jq$ with i, j monomorphisms and p, q strong epimorphisms, then there is an isomorphism c making the following diagram commute.*

$$\begin{tikzcd} {} \arrow[r, "q"] \arrow[d, "p"'] & {} \arrow[d, "j"] \arrow[dl, "c"'] \\ {} \arrow[r, "i"'] & {} \end{tikzcd}$$

□

Suppose $f: A \to B$ has canonical factorization ip with $p: A \to C$ and $i: C \to B$, then as a result of Proposition 3.3, we are able to refer to C as the *image* of f.

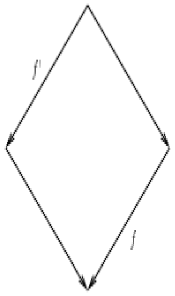
Finally, following [7], we are able to define a regular category.

Definition 3.4 A *regular category* is a category in which:

(1) Every arrow has a canonical factorization;

(2) All finite limits exist;

(3) Strong epimorphisms are *stable* — i.e. in every pullback diagram pictured below, f is a strong epimorphism implies that f' is a stron epimorphism.



Some examples of regular categories:

- **Set** — the category of sets and functions. The strong epimorphisms (which are all of the epimorphisms) are the surjective functions.
- **Mon** — the category of monoids and monoid homomorphisms. The strong epimorphisms are the surjective monoid homomorphisms. These are not, however, all the epimorphisms.

  An example (cited in [1]) of an epimorphism in **Mon** which is not surjective is the canonical inclusion (as multiplicative monoids) of the integers into the rationals.

For the remainder of Section 3.1, **C** will denote a regular category.

3.1.2 The bicategory Rel(**C**)

We define the bicategory Rel(**C**), which in the case that **C** = **Set**, has sets as objects and an arrow from the set A to the set B will be a subset of $A \times B$.

Let **C** be a regular category and let A, B be objects of **C**. Following [6], a *relation* $R: A \to B$ in **C** is a span from A to B, such that if $S: A \to B$ is any other span, then there is at most one 2-cell in Span(**C**) from S to R. It is an easy exercise to see that for any relation $R = (f, R, g)$, the arrow $\langle f, g\rangle: R \to A \times B$ is monic. Conversely, if $i: R \to A \times B$ is monic, then $(\pi_A i, R, \pi_B i)$ is a relation.

Definition 3.5 Every span $S = (f, S, g)$, defines a relation: let $S \xrightarrow{p} Q \xrightarrow{i} A \times B$ be the canonical factorization of $\langle f, g\rangle: S \to A \times B$. Then $Q = (\pi_A i, Q, \pi_B i)$ is a relation and p is a 2-cell from S to Q. Such Q is unique up to isomorphism, hence we call it *the relation defined by* S.

The bicategory Rel(**C**) of relations in **C** is defined as follows:

Rel–I The objects of Rel(**C**) are the objects of **C**.

Rel–II The arrows of Rel(**C**) are the relations in **C**. The 2-cells are the 2-cells of Span(**C**). That is, for any objects A, B of **C**, the category Rel(**C**)(A, B) is the full subcategory of Span(**C**)(A, B) whose objects are the relations.

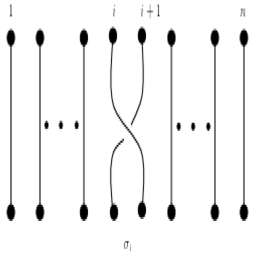
Rel–III The composite of the relations $R: A \to B$ and $S: B \to C$ is obtained by composing as spans and then taking the relation defined by the composite span.

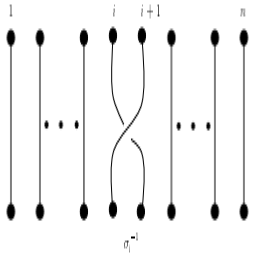
The remaining details of this definition are left to the reader. As one would hope, if **C** = **Set**, then the composite of R and S in Rel(**C**) is the set $\{(a, c) \mid (a, b) \in R, (b, c) \in S,$ for some $b \in B\}$.

An Introduction to Vines

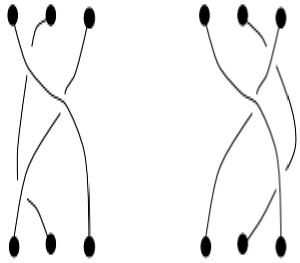
In this section, we give an informal introduction to Lavers' theory of [15], and from this basis, develop the category **Vine** of vines and inves the category of concrete vine monoids. The reader is assumed to be fa with Artin's theory of braids [2]. The category **Vine** is not regular (si is not finite complete) and so provides us with an example of a category which we can construct **Conc**(**Vine**) but not Div(**Reps**(**Vine**)).

In the following discussion $[n]$ will refer to the set $\{1, 2, \ldots, n\}$, wit usual total order. Fix $n > 0$. An n-braid is a set of n arcs in $\mathbb{R}^3$ fro n initial nodes $I(n) = \{(i, 0, 1) \mid i \in [n]\}$ to the n terminal nodes T $\{(i, 0, 0) \mid i \in [n]\}$, each of which is strictly decreasing in the z-coord Every braid may be regarded as a product of the $2(n-1)$ generators, pic below for $1 \leq i \leq n-1$.

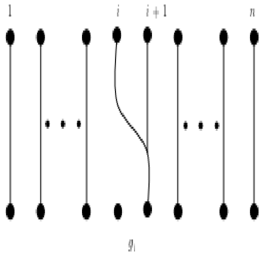




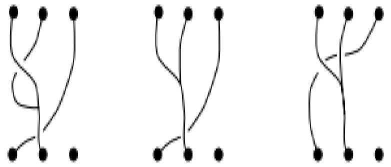
Two braids are regarded as equivalent if they can be deformed, one into the other, by a homotopy as in the following diagram.



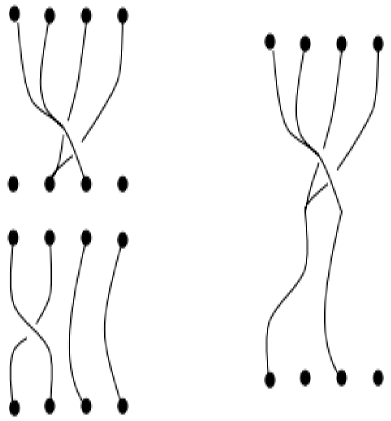
Rather than give a formal definition for vines, we rely on the reader's intuition about braids and some illuminating examples. A *vine* from $I(n)$ to $T(n)$ may be written as a product of the generators σ_i and σ_i^{-1} as pictured above, together with the generators g_i shown below, where $1 \leq i \leq n-1$.



As a result, the strings of a vine may merge, but not separate. Among the homotopy equivalence relations we have $\sigma_i g_i = \sigma_i^{-1} g_i = g_i$, meaning that strings can twist and untwist about a join. Some examples of equivalent vines follow.



(Note: The word 'vine' may be taken to mean either the set of arcs or, the homotopy equivalence class of the set of arcs, depending on the context.) As with braids, composition is given by concatenation and shrinkage, and is a well-defined and associative operation on homotopy equivalence classes of vines. Note that if a string in the concatenation is not connected to any initial node, it is homotopically shrunk to a point. An example of composition of vines follows.



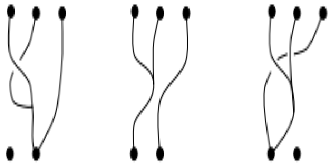
The set of equivalence classes of vines is a monoid with the sam element as the braid group — a string connects the ith initial node terminal node with no intertwining or joining of the strings. Defining for the monoid of all vines on n nodes are given in [15].

Lavers' vines always go from n initial nodes to n terminal nodes. here the straightforward generalization that a vine may go from $I(n$ where n is not necessarily equal to m — it consists of n strings (to each of the n initial nodes, each string ending at a node in $T($ will be known as an $n \to m$ vine. Composition of vines is then onl when the number of terminal nodes of the first vine is equal to th of initial nodes of the second. This generalization permits us to (category **Vine** whose objects are the natural numbers $0, 1, 2, \ldots$ a arrows from n to m are precisely the (homotopy equivalence classes o vines.

Definition 13.1 If v is an $n \to m$ vine, let $\overline{v}$ denote the map $[n]$ – maps i to k if the ith string meets the kth terminal node $(k, 0, 0)$ The map $\overline{v}$ will be called the *function associated with* v.

Proposition 13.2 *There is a functor* **Vine** $\to$ **Set** *which takes* n ; *the* $n \to m$*-vine* v *to* $\overline{v}$.

A *simple vine* is one which is homotopic to a vine in which tl never intertwine. Some simple vines are pictured below.



A *braid* is a vine which has no joins, that is to say, a vine v such that $\overline{v}$ is injective.

Proposition 13.3 *The simple* $n \to n$*-vines form a monoid isomorphic to the monoid of order-preserving transformations of* $\{1, 2, \ldots, n\}$. *The isomorphism is given by mapping* v *to* $\overline{v}$*, the function associated with* v. □

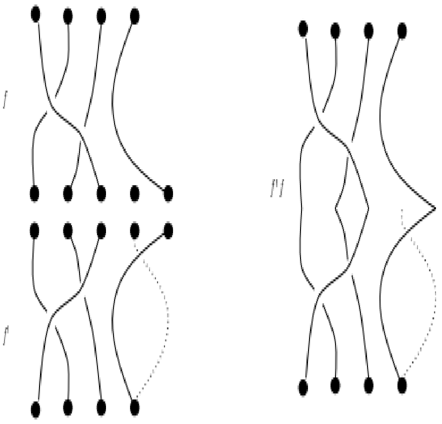
The following theorem is a direct consequence of [15, Theorem 2].

Theorem 13.4 *Every vine* $u : m \to n$ *can be written* $u = vb$ *where* b *is a* $m \to m$*-braid and* $v : m \to n$ *is a simple vine.* □

Lemma 13.5 *Every braid* $n \to p, n > 0$ *has a left inverse.*

Proof. Let $f : n \to p$ be a braid. Consider the picture obtained by reflection of f in the $x-y$ plane. This may not be a vine since there is not necessarily an arc emanating from every node in the top plane of the reflection. Construct a vine by adding arcs emanating from top nodes which are not already at the top of an arc. The bottom nodes at which they terminate may be chosen arbitrarily. Call the vine so obtained f'. Then it is clear that $f'f = 1_n$. □

An example of the construction of Lemma 13.5 is pictured below.



Theorem 13.6 *Braids are the monic arrows of* **Vine**.

Proof. By Lemma 13.5 we see that braids $n \to p, n > 0$ are certainly monic. Further, since there are is only one arrow into 0, every braid $0 \to p$ is monic, thus all braids are monic. Conversely, suppose the vine $u : m \to n$ is not a braid. Write (by Theorem 13.4) $u = vb$ where b is a $m \to m$-braid and $v : m \to n$ is a simple vine. Let b^{-1} be the (group) inverse of b. Since vb has joins, there must be consecutive nodes indexed by $i, i+1$ in $I(m)$ such that $\overline{v}(i) = \overline{v}(i+1)$. Let $s : 2 \to m$ be a braid which connects $1 \in I(2)$ to $i \in T(m)$ and $2 \in I(2)$ to $i+1 \in T(m)$. Let $t : 2 \to m$ be a braid which connects $1 \in I(2)$ to $i+1 \in T(m)$ and $2 \in I(2)$ to $i \in T(m)$. Then it is clear that

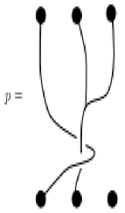
$$\begin{aligned} ub^{-1}s &= vbb^{-1}s \\ &= vs \\ &= vt \\ &= ub^{-1}t. \end{aligned}$$

But since b^{-1} is monic, $b^{-1}s \neq b^{-1}t$, proving that u is not monic. □

The category **Vine** has initial object 0 and terminal object 1. Thus we may construct the category **Conc**(**Vine**). An object of **Conc**(**Vine**) is called a *vine monoid*.

13.1 An arrow of **Conc**(**Vine**).

Consider the vine monoid M_3 on 3 strings generated by p as in the following diagram.

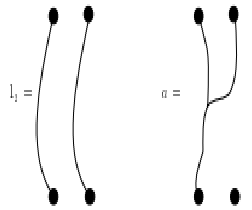


The single generator p of M_3.

ISSN 2454 – 5X

http://www.ijmert.com
Vol. 15 Issue 4, Dec 2023

This is clearly infinite, and does not contain the two element monoi (shown below) as a submonoid.



The vine monoid N_2 with elements $\{1, a\}$.

However N_2 may certainly be found as a divisor by the division $((\natural, 1_2), M_2$ where $M_2(p)$ is defined to be the 2-vine a, η is $\sigma_1^2 : 2 \rightarrow 3$ (shown in F 2), and $\natural : M \rightarrow N$ is the natural quotient. To see that $(1_M, \eta)$ is an of **Reps**(**Vine**) simply notice that $\eta M_2(p) = M_3(p)\eta$ as in Figure 2. The arrow η is not the simplest arrow from N_2 to M_3 in **Conc**(**Vine**) deliberately chose a less obvious arrow in order to show the robustness c intuition with respect to the possibilities allowed by the definition.

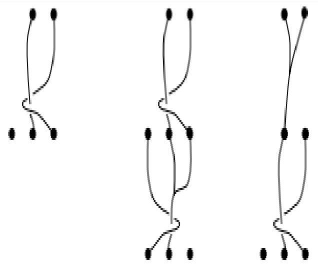


Fig. 2. From left to right: η, $M_3(p)\eta$ and $\eta M_2(p)$.

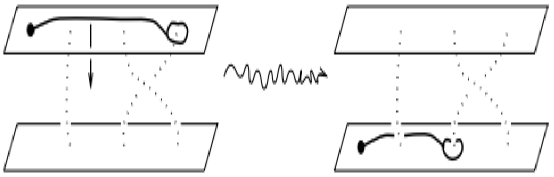
13.2 *From* Vine *to* FreeGrp

Let $\{x_1, x_2, \ldots\}$ be an infinite alphabet. Denote by F_n the free group on the generators $\{x_1, \ldots x_n\}$. Let **FreeGrp** be the category whose objects are F_n, $n \geq 0$, and whose arrows are all group homomorphisms between them. The trivial group is free of rank 0, so that it is initial and terminal in both **FreeGrp** and **FreeGrp**^{op}. We can therefore speak of the category **Conc**(**FreeGrp**^{op}).

Firstly, it is easy to see that

Proposition 13.7 *Surjective maps in* **FreeGrp** *are epimorphic.* $\square$

Artin's famous representation theorem states that every $n \rightarrow n$-braid defines an automorphism on the free group of rank n. This proceeds by identifying the free group with generators $\{x_1, \ldots, x_n\}$ with the homotopy group of the plane with a puncture at each of the coordinates of $I(n)$. Then the braid acts on the fundamental group as in the following diagram.



Via the observation that this action can be reversed, we formulate the following representation theorem for vines. It is the first part of [15, Theorem 6].

Theorem 13.8 *The monoid* V_n *of all* $n \rightarrow n$ *vines has a faithful contravariant representation as a monoid of endomorphisms of a free group* F_n *of rank* n. *The representation is induced by a mapping* $\zeta : V_n \rightarrow End(F_n)$ *defined by*

$$\zeta(s_i) : x_j \longmapsto \begin{cases} x_j & if\, j \notin \{i, i+1\} \\ x_i x_{i+1} x_i^{-1} & if\, j = i \\ x_i & if\, j = i+1 \end{cases}$$

and

$$\zeta(g_i) : x_j \longmapsto \begin{cases} x_j & if\, j \notin \{i, i+1\} \\ 1 & if\, j = i \\ x_i x_{i+1} & j = i+1 \end{cases} .$$

$\square$

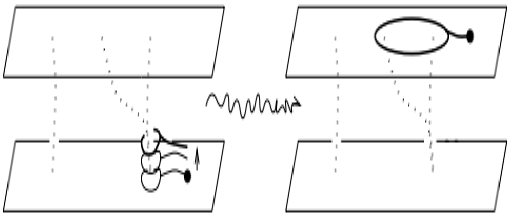
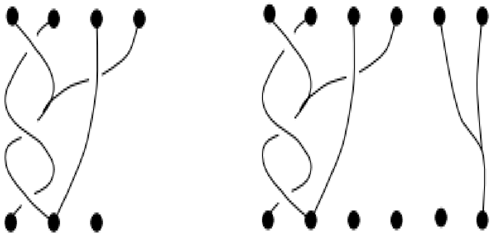


Fig. 3. The contravariant action of a vine on a generator of the fundamental group of the 3-punctured plane.

This representation theorem is now used to define a functor Γ: **FreeGrp**^{op} which is faithful and preserves monos, and hence induces functor **Conc**(**Vine**) $\to$ **Conc**(**FreeGrp**^{op}).

Let $m < q$ be positive integers. Define $p_{q,m}: F_q \to F_m$ to be the h phism which acts identically on the generator x_i if $i \leq m$ and maps otherwise. Let $i_{m,q}: F_m \to F_q$ be the natural injection.

Let $v: m \to n$ be a vine and let q be greater than both m and n $\gamma_v^q: q \to q$ to be the vine which is the same as v from the first m starti to the first n terminal nodes but which has a simple vine from the nodes numbered $m+1$ through q, to the q^{th} terminal node, as in the diagram.



The vines $v: 4 \to 3$ and $\gamma_v^6: 6 \to 6$.

When there can be no confusion as to the value of q, we will oft γ_v instead of γ_v^q.

Definition 13.9 For each object n of **Vine** define $\Gamma(n)$ to be F_n in **F** Let $v: m \to n$ be a vine, and let q be any integer greater than both Then $\Gamma(v)$ is the homomorphism $p_{q,m}\zeta(\gamma_v^q)i_{n,q}: F_n \to F_m$.

This definition is independent of q and corresponds to the action by Figure 4.

The following proposition is clear.

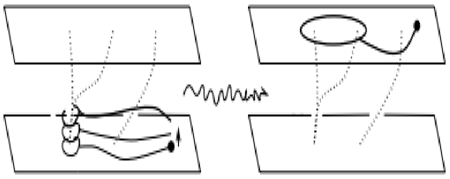


Fig. 4. The geometric interpretation of the action of a vine $v: 3 \to 2$ as a homomorphism $\Gamma(v): F_2 \to F_3$.

Proposition 13.10 *Let* $k \xrightarrow{u} m \xrightarrow{v} n$ *be a diagram in* **Vine**, *and let* q *be greater than* k, m *and* n. *Then* $\gamma_v^q \gamma_u^q = \gamma_{vu}^q$. ☐

Lemma 13.11 Γ *is a functor* **Vine** $\to$ **FreeGrp**^{op}.

Proof. It is clear that $\Gamma(1_n) = 1_{F_n}$. We just need to show that for each diagram $k \xrightarrow{u} m \xrightarrow{v} n$ in **Vine**, $\Gamma(vu) = \Gamma(u)\Gamma(v)$ in **FreeGrp**. Fix $q > \max\{k, m, n\}$. By definition $\Gamma(vu) = p_{q,k}\zeta(\gamma_{vu})i_{n,q}$, while $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_u)i_{m,q}p_{q,m}\zeta(\gamma_v)i_{n,q}$. Now the image of $\zeta(\gamma_v)i_{n,q}$ lies in the subgroup of F_q generated by $\langle x_1, \ldots, x_m \rangle$ upon which $p_{q,m}$ acts identically, so that $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_u)\zeta(\gamma_v)i_{n,q}$.

By Theorem 13.8, ζ is a contravariant monoid homomorphism on objects so that $\zeta(\gamma_u)\zeta(\gamma_v) = \zeta(\gamma_v\gamma_u)$. But by Proposition 13.10 $\gamma_v\gamma_u = \gamma_{vu}$ so that $\Gamma(u)\Gamma(v) = p_{q,k}\zeta(\gamma_{vu})i_{n,q} = \Gamma(vu)$ as required. ☐

Lemma 13.12 *The functor* Γ *is faithful.*

Proof. Let $u, v: m \to n$ be vines, and let q be greater than both m and n. F
the remainder of this proof p will denote $p_{q,m}$ and i will denote $i_{n,q}$. Suppos
$\Gamma(u) = \Gamma(v)$. Then for all generators x_j of F_n, $p\zeta(\gamma_v)i(x_j) = p\zeta(\gamma_u)i(x_j$
whence, for each generator x_j of F_q with $j \leq n$,

(4) $p\zeta(\gamma_v)(x_j) = p\zeta(\gamma_u)(x_j)$.

If $\zeta(\gamma_v)$ were not equal to $\zeta(\gamma_u)$, then by the construction of γ they woul
only differ on some generators $x_i, i \leq n$. In combination with equation (4
this yields $p\zeta(\gamma_v)(x_j) = p\zeta(\gamma_u)(x_j)$ for all $j \leq q$.

Suppose $\zeta(\gamma_v) \neq \zeta(\gamma_u)$, then for some $x_i, i \leq n$, $\zeta(\gamma_v)(x_i) \neq \zeta(\gamma_u)(x_i$
while both $\zeta(\gamma_v)(x_i)$ and $\zeta(\gamma_u)(x_i)$ are elements of the subgroup $\langle x_1, \ldots, x_m$
of F_q, by the construction of γ. But p is injective on this subgroup, therefo
$\zeta(\gamma_v)(x_i) = \zeta(\gamma_u)(x_i)$ for all generators x_i of F_q. By Theorem 13.8, this impli
that $\gamma_v = \gamma_u$ and hence, by the construction of γ, $v = u$.

Lemma 13.13 *If v is a monic vine then $\Gamma(v)$ is a surjective homomorphism
and therefore a monic arrow of* **FreeGrp**^{op}.

Proof. Since a monic vine is a braid (b, say), any generating loop x_i at th
top may be "pushed down" via Artin's representation to get a loop σ at th
bottom. When σ is acted upon by $\Gamma(b)$ we get x_i back again. Thus, ever
generating loop is in the image of $\Gamma(b)$ as required.

References

[1] Adámek, J., H. Herrlich, and G. E. Strecker, 'Abstract and concrete cat
(Wiley-Interscience, 1990).

[2] Artin, E., *Theory of braids*, Annals of Mathematics (2), **48** (1947), 101

[3] Bénabou, J., 'Introduction to bicategories', Lecture Notes in Mathema
(Springer-Verlag, Berlin, 1967).

[4] Cannon, J., and D. F. Holt, *Computing chief series, composition se
socles in large premutation groups*, (in preparation).

[5] Cannon, J., and D. F. Holt, *Computing Sylow subgroups in permutation*
(in preparation).

[6] Carboni, A., S. Kasangian and R. Street, *Bicategories of spans and r*
Journal of Pure and Applied Algebra **33** (1984), 259–267.

[7] Cassidy, C., M. Hébert and G. M. Kelly, *Reflective subcategories, localizations and factorization systems*, J. Austral. Math. Soc. (Series A) **38** (1985), 287–329.

[8] Ehrig, H. et. al., 'Universal theory of automata', (Teubner Studienbücher, Stuttgart, 1974).

[9] Eilenberg, S., 'Automata, languages and machines', Vol. A, (Academic Press, New York, 1974).

[10] Goguen, J. A., *Realization is universal*, Mathematical Systems Theory, **6**, (1972/73), 359–374.

[11] Higgins, P. M., *Pseudovarieties generated by transformation semigroups*, (submitted).

[12] Howie, J. M., 'An introduction to semigroup theory', (Academic Press, London, 1976).

[13] Kelly, G. M., *Monomorphisms, epimorphisms and pull-backs*, J. Austral. Math. Soc. **9** (1969), 124–142.

[14] Lallement,G.,'Semigroups and combinatorial applications', (Wiley-Interscience, New-York, 1979).

[15] Lavers, T. G., 'The theory of vines', Communications in Algebra **25**(4) (1997), 1257–1284.

[16] Mac Lane, S., ‘Categories for the worki 1971).

[17] Nehaniv, C. L., *From relation to e* *transformation semigroups*, Journal of P 75–87.

[18] Repnitskiĭ, V. B., and M. V. Volkov, *pseudovariety $\mathcal{O}$*, Department of Math University, 620083, Ekaterinburg, Russia.

[19] Schumacher, D., and R. Street, *Som* Communications in Algebra, 16 (11), 231

[20] Solomon, A., *Catalan monoids, monoid* *presentations*, Semigroup Forum **53** (199

[21] Tilson, B., *Monoid kernels, semidirect pr* in: J. Rhodes, ed., Monoids and Semigrou the Berkeley Workshop in Monoids, (Wor

[22] Tilson, B., *Categories as algebra: an* *monoids*, Journal of Pure and Applied Al